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ON A CONJECTURE OF GOODEARL: JACOBSON RADICAL NON-NIL ALGEBRAS OF GELFAND-KIRILLOV DIMENSION 2

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Abstract. For an arbitrary countable field, we construct an associative algebra that is graded, generated by finitely many degree-1 elements, is Jacobson radical, is not nil, is prime, is not PI, and has Gelfand-Kirillov dimension two. This refutes a conjecture attributed to Goodearl.

1. Introduction

Consider an algebra R over a field $\mathbb{K}$, generated by a finite-dimensional subspace V. The *Gelfand-Kirillov dimension*, or *GK-dimension*, of R is the infimal d such that $\dim(V + V^2 + \cdots + V^n)$ grows slower than n^d as $n \to \infty$. For example, $\mathbb{K}[t_1, \dots, t_d]$ has GK-dimension d. Which constraints does an associative algebra of finite Gelfand-Kirillov dimension have to obey? For example, if R is a group ring, then the group has polynomial growth, so is virtually nilpotent by Gromov's celebrated theorem [5], so R is noetherian. For elementary properties of the Gelfand-Kirillov dimension, see [7].

However, various flexible constructions have produced quite exotic examples of finitely generated associative algebras (*affine algebras* in the sequel) of finite GK-dimension [3], and it has been hoped at least that algebras of GK-dimension 2 would enjoy some sort of classification — algebras of GK-dimension < 2 are well understood, and are essentially polynomials in at most one variable, by Bergman's gap theorem [4], and graded domains of GK-dimension 2 are essentially twisted coördinate rings of projective curves [1].

An element x in a ring R is *quasi-regular* if there exists $y \in R$ with $x + y + xy = 0$. This happens, for instance, if x is nilpotent (take $y = -x + x^2 - x^3 + \cdots$). Conversely, if R is graded, then homogeneous quasi-regular elements are nilpotent. The *Jacobson radical* $J(R)$ of R is the largest ideal all of whose elements are quasi-regular. A ring is *radical* if it is equal to its Jacobson radical; note then, in particular, that it may not contain a unit (in fact, not even a non-trivial idempotent: $x^2 = x, -x + y - xy = 0 \Rightarrow -x^2 + xy - x^2y = -x^2 = -x = 0$).

A typical result showing the connection between nillity and the structure of the Jacobson radical is: R is artinian, then $J(R)$ is nilpotent. The following structural result was expected:

Conjecture (Goodearl, [3, Conjecture 3.1]). *If R is an affine algebra of GK-dimension 2, then its Jacobson radical $J(R)$ is nil.*

We disprove this conjecture, by constructing for every countable field $\mathbb{K}$ an algebra R over $\mathbb{K}$, which is

- graded by the natural numbers;

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- generated by finitely many degree-1 elements;
- prime;
- of Gelfand-Kirillov dimension 2;
- equal to its Jacobson radical;
- not PI (i.e. does not satisfy a polynomial identity);
- not nil.

Our strategy is to adapt a construction of the first author, see [13], by showing that it may yield non-nil algebras. Some tools are also borrowed from the second author's paper [2]; however, the construction given there is not correct, and indeed not not yield a radical algebra. One of the goals of this paper is therefore to give a correct solution to the problem raised by Goodearl.

2. The construction

We begin by constructing the following algebra P; the proof of this theorem will be split over the next three sections.

Theorem 2.1. *Over every countable field $\mathbb{K}$ of characteristic zero, there exists a radical algebra P, such that the polynomial ring $P[X]$ is not radical.*

Moreover, P may be chosen to have Gelfand-Kirillov dimension two, be $\mathbb{N}$-graded and generated by two elements of degree one.

We then show that a sufficiently large ring of matrices over such a P is not nil:

Proposition 2.2. *Let P be a radical algebra such that the polynomial ring $P[X]$ is not radical. Then there is a natural number n such that the algebra $M_n(P)$ of n by n matrices over P is not nil.*

Proof. Suppose that P is radical and that, for every $n \in \mathbb{N}$, the ring $M_n(P)$ is nil. Write $R = P[X]$ and $\mathscr{I} = XR$; we will deduce that R is radical. Observe that $M_n(XP)$ is nil for all $n \in \mathbb{N}$, and $\mathscr{I} = XP + (XP)^2 + \cdots$; therefore, by [12, Theorem 1.2], the ring $\mathscr{I}$ is radical. Notice then that $\mathscr{I}$ is an ideal in R, and $R/\mathscr{I} = P$ is radical. Now, if both $\mathscr{I}$ and $R/\mathscr{I}$ are radical, then so is R. □

Lemma 2.3. *Let R be a non-nil ring. Then there exists a quotient $R/\mathscr{I}$ that is non-nil and prime. If R is graded, then $R/\mathscr{I}$ may also be taken to be graded.*

Proof. Let $a \in R$ be non-nilpotent. Let $\mathscr{I}$ be a maximal ideal in R subject to being disjoint with $\{a^n : n = 1, 2, \dots\}$. Then $R/\mathscr{I}$ is still not nil. Consider ideals $\mathscr{P}, \mathscr{Q} \supsetneq \mathscr{I}$ with $\mathscr{P}\mathscr{Q} \subseteq \mathscr{I}$. By maximality of $\mathscr{I}$, we have $a^n \in \mathscr{P}$ and $a^m \in \mathscr{Q}$ for some $m, n \in \mathbb{N}$; but then $a^{m+n} \in \mathscr{I}$, a contradiction. Therefore, $R/\mathscr{I}$ is prime.

If R is graded, let $\mathscr{I}$ be a maximal *homogeneous* ideal subject to being disjoint with $\{a^n : n = 1, 2, \dots\}$. We claim that $\mathscr{I}$ is a prime ideal in R. Suppose the contrary; then there are elements $p, q \notin \mathscr{I}$ such that such that $prq \in \mathscr{I}$ for all $r \in R$. Write $p = p_1 + \cdots + p_d$ and $q = q_1 + \cdots + q_e$ in homogeneous components, and let p_i and q_j denote those summands, for minimal i, j, that do not belong to $\mathscr{I}$.

By assumption, $prq \in \mathscr{I}$ for all homogeneous $r \in R$ (say of degree k); so, by considering the component of degree $i+k+j$ of prq, we see that $p_i r q_j$ belongs to $\mathscr{I}$ for all homogeneous $r \in R$ (because $\mathscr{I}$ is graded), whence $p_i r q_j \in \mathscr{I}$ for all $r \in R$.

Let now $\mathscr{P}$ be the ideal generated by p_i and $\mathscr{I}$; and, similarly, let $\mathscr{Q}$ be the ideal generated by q_j and $\mathscr{I}$. Then, by maximality of $\mathscr{I}$, we have $a^n \in \mathscr{P}$ and $a^m \in \mathscr{Q}$ for some $m, n \in \mathbb{N}$; but then $a^{m+n} \in \mathscr{P}\mathscr{Q} \subseteq \mathscr{I}$, a contradiction. Therefore, $R/\mathscr{I}$ is prime. □

Combining these results, we deduce:

Corollary 2.4. *Over any countable field $\mathbb{K}$, there exists a non-nil non-PI radical prime algebra R, of Gelfand-Kirillov dimension two, $\mathbb{N}$-graded and generated by finitely many elements of degree one.*

Proof. Let P be as in Theorem 2.1. By Proposition 2.2, the ring $R_0 = M_n(P)$ is radical and non-nil for n large enough. By Lemma 2.3, some quotient R of R_0 is radical and prime. Because P is radical, its ring of matrices R_0 is also radical, and so is its quotient R. Because P has GK-dimension ≤ 2, so do R_0 and R. If R has GK-dimension < 2, it would have dimension ≤ 1 by Bergman's gap theorem [4], so would be finitely generated as a module over its centre by [10], so R's radical would be nilpotent, a contradiction; therefore, R has GK-dimension exactly 2.

Since P is generated by 2 elements of degree 1, the rings R_0 and R are generated by finitely many elements of degree 1 (the elementary matrices).

Finally, R is not PI; indeed, by the Razmyslov-Kemer-Braun theorem [6, §2.5], if R were PI then its radical would be nilpotent. □

3. Notation and previous results

Our notation closely matches that of [13]. In what follows, $\mathbb{K}$ is a countable field and A is the free associative $\mathbb{K}$-algebra in three non-commuting indeterminates x, y, z. The set of monomials in $\{x, y\}$ is denoted by M and, for $n \geq 0$, the set of monomials of degree n is denoted by $M(n)$. In particular, $M(0) = \{1\}$ and for $n \geq 1$ the elements in $M(n)$ are of the form $x_1 \cdots x_n$ with $x_i \in \{x, y\}$. The *augmentation ideal* of A, consisting of polynomials without constant term, is denoted by $\bar{A}$.

The $\mathbb{K}$-subspace of A spanned by $M(n)$ is denoted by $A(n)$, and elements of $A(n)$ are called *homogenous polynomials of degree n*. More generally, if S is a subset of A, then its homogeneous part $S(n)$ is defined as $S \cap A(n)$.

The *degree*, $\deg f$, of $f \in A$, is the least $d \geq 0$ such that $f \in A(0) + \cdots + A(d)$. Any $f \in A$ can be uniquely written in the form $f = f_0 + f_1 + \cdots + f_d$, with $f_i \in A(i)$. The elements f_i are the *homogeneous components* of f. A (right, left, two-sided) ideal $\mathscr{I}$ of A is *homogeneous* if, for every $f \in \mathscr{I}$, all its homogeneous components belong to $\mathscr{I}$.

Lemma 3.1 ([13, Lemma 6])**.** *Let $\mathbb{K}$ be a countable field, and let $\bar{A}$ be as above. Then there exists a subset $Z \subset \{5, 6, \dots\}$, and an enumeration $\{f_i\}_{i \in Z}$ of $\bar{A}$, such that*

$$i > 3^{2deg(f_i)+2}(\deg(f_i) + 1)^2 \text{ for all } i \in Z.$$

Define the sequence $e(i) = 2^{2^{2^{2^i}}}$, and set

$$S = \bigcup_{i \geq 5} \{e(i) - i - 1, e(i) - i, \dots, e(i) - 1\}.$$

Lemma 3.2 ([13, Theorem 9])**.** *Let Z and $\{f_i\}_{i \in Z}$ be as in Lemma 3.1. Fix $m \in Z$, and set $w_m = 2^{e(m)+2}$. Then there is a two-sided ideal $\mathscr{P}_m \leq \bar{A}$ such that*

- *the ideal $\mathscr{P}_m$ is generated by homogeneous elements of degrees larger than $10w_m$;*
- *there exists $g_m \in \bar{A}$ such that $f_m - g_m + f_m g_m \in \mathscr{P}_m$;*
- *there is a linear $\mathbb{K}$-space $F_m \subseteq A(2^{e(m)})$ such that $\mathscr{P}_m \subseteq \sum_{k=0}^{\infty} A(w_m k) F_m A$ and $dim_{\mathbb{K}}(F_m) < m$.*

Lemma 3.3 ([13, Theorem 10])**.** *Let Z and F_m be as in Lemma 3.2. There are $\mathbb{K}$-linear subspaces $U(2^n)$ and $V(2^n)$ of $A(2^n)$ such that, for all $n \in \mathbb{N}$,*

1. $\dim_{\mathbb{K}} V(2^n) = 2$ *if $n \notin S$;*
2. $\dim_{\mathbb{K}} V(2^{e(i)-i-1+j}) = 2^{2^j}$*, for all $i \geq 5$ and all $j \in \{1, \dots, i-1\}$;*
3. $V(2^n)$ *is spanned by monomials;*

(4) *$F_i \subseteq U(2^{e(i)})$ for every $i \in Z$;*
(5) *$V(2^n) \oplus U(2^n) = A(2^n)$;*
(6) *$A(2^n)U(2^n) + U(2^n)A(2^n) \subseteq U(2^{n+1})$;*
(7) *$V(2^{n+1}) \subseteq V(2^n)V(2^n)$;*
(8) *if $n \notin S$ then there are monomials $m_1, m_2 \in V(2^n)$ such that $V(2^n) = \mathbb{K}m_1 + \mathbb{K}m_2$ and $m_2 A(2^n) \subseteq U(2^{n+1})$.*

4. New results

Consider the polynomial ring $A[X]$ in an indeterminate X. Consider the elements $(x + Xy)^n$. Write

$$w(n,i) = \sum_{\substack{m \in M(n) \\ \deg_y m = n-i, \deg_x m = i}} m,$$

and observe that $(x + Xy)^{2^n} = \sum_{i=0}^{2^n} w(2^n, 2^n - i)X^i$. Let $W(n)$ denote the linear span of all $w(n,i)$ with $i \in \{0, \dots, n\}$.

We extend the results of the previous section by imposing additional conditions on the $U(n)$ and $V(n)$ constructed in Lemma 3.3. Throughout this section, we use the notation

$$T(2^{n+1}) = A(2^n)U(2^n) + U(2^n)A(2^n).$$

Proposition 4.1. *There exist subspaces $U(2^n), V(2^n) \subseteq A(2^n)$ satisfying all assumptions from Lemma 3.3, with the additional property that*

(9) *for all $n \in \mathbb{N}$, if $i \in \mathbb{N}$ be such that $\{n, n-1, \dots, n-i\} \subset S$, then*

$$\dim_{\mathbb{K}}(W(2^n) + U(2^n)) \geq \dim_{\mathbb{K}} U(2^n) + 2 + i;$$

(10) *$z \in U(2^0) = U(1)$.*

Lemma 4.2. *If $\dim_{\mathbb{K}}(W(2^n) + U(2^n)) \geq \dim_{\mathbb{K}} U(2^n) + 2$ and $m_1, m_2 \in V(2^n)$ are linearly independent, then there exists $h \in \{1, 2\}$ such that*

$$\dim_{\mathbb{K}}(W(2^{n+1}) + T(2^{n+1}) + m_h V(2^n)) \geq \dim(T(2^{n+1}) + m_h V(2^n)) + 2.$$

Proof. Let $i \geq 0$ be minimal such that $w(2^n, i)$ does not belong to $U(2^n)$, and let $j > i$ be minimal such that $w(2^n, j)$ does not belong to $U(2^n) + \mathbb{K}w(2^n, i)$. By the inductive assumption such elements can be found. By permuting m_1 and m_2 if necessary, we may assume that $w(2^n, i)$ is not a multiple of m_2, and we choose $h = 2$. We have

$$w(2^{n+1}, 2i) = \sum_{k=-i}^{i} w(2^n, i+k)w(2^n, i-k),$$

and either

1. $k = 0$,
or 2. $k < 0$, in which case $w(2^n, i+k) \in U(2^n)$,
or 3. $k > 0$, in which case $w(2^n, i-k) \in U(2^n)$.

Consequently, we get

$$w(2^{n+1}, 2i) \equiv w(2^n, i)w(2^n, i) \mod T(2^{n+1}). \tag{1}$$

Consider now

$$w(2^{n+1}, i+j) = \sum_{k=-i}^{j} w(2^n, i+k)w(2^n, j-k);$$

then either

1. $k < 0$, in which case $w(2^n, i+k) \in U(2^n)$,

or 2. $0 < k < j - i$, in which case $w(2^n, i+k) \in U(2^n) + \mathbb{K}w(2^n, i)$ and $w(2^n, j - k) \in U(2^n) + \mathbb{K}w(2^n, i)$,
or 3. $k = 0$ or $k = j - i$,
or 4. $k > j - i$, in which case $w(2^n, j - k) \in U(2^n)$.

Consequently, we get

$$(2) \quad w(2^{n+1}, i+j) \equiv w(2^n, i)w(2^n, j) + w(2^n, j)w(2^n, i) \mod T(2^{n+1}) + \mathbb{K}w(2^n, i)w(2^n, i).$$

Recall now that we have

$$w(2^n, i) \equiv t_{i1}m_1 + t_{i2}m_2 \mod U(2^n), \qquad w(2^n, j) \equiv t_{j1}m_1 + t_{j2}m_2 \mod U(2^n)$$

for some $t_{i1}, t_{i2}, t_{j1}, t_{j2} \in \mathbb{K}$. Furthermore, $t_{i1} \neq 0$, and the vectors (t_{i1}, t_{i2}) and (t_{j1}, t_{j2}) are linearly independent over $\mathbb{K}$. Write $Q = T(2^{n+1}) + m_2V(2^n)$, so that Q contains m_2m_2 and m_2m_1.

It follows from (1) that $w(2^{n+1}, 2i) \equiv t_{i1}^2 m_1m_1 + t_{i1}t_{i2}m_1m_2 \mod Q$; and, because $t_{i1} \neq 0$, we have $w(2^{n+1}, 2i) \notin Q$.

Similarly, from (2) we get $w(2^{n+1}, i+j) \equiv 2t_{i1}t_{j1}m_1m_1 + (t_{j1}t_{i2} + t_{i1}t_{j2})m_1m_2 \mod Q + \mathbb{K}w(2^{n+1}, 2i)$; and, because the vectors (t_{i1}, t_{i2}) and (t_{j1}, t_{j2}) are linearly independent, so are $(2t_{i1}t_{j1}, t_{j1}t_{i2} + t_{i1}t_{j2})$ and $(t_{i1}^2, t_{i1}t_{i2}) = t_{i1}(t_{i1}, t_{i2})$, so we have $w(2^{n+1}, i+j) \notin Q + \mathbb{K}w(2^{n+1}, 2i)$.

We then get $\dim_{\mathbb{K}}(W(2^{n+1}) + Q) \geq \dim_{\mathbb{K}} Q + 2$ as required. □

Lemma 4.3. $\dim_{\mathbb{K}}(W(2^{n+1}) + T(2^{n+1})) \geq \dim_{\mathbb{K}}(W(2^n) + T(2^n)) + 1$.

Proof. Let there be $k_1, k_2, \ldots, k_j \in \mathbb{N}$ such that

$$w(2^n, k_1), w(2^n, k_2), \ldots, w(2^n, k_j)$$

are linearly independent modulo $T(2^n)$. We may assume that the sequence $(k_1, \ldots, k_j)$ is minimal with this property in the lexicographical ordering. We claim that the elements $w(2^{n+1}, 2k_j)$ and $w(2^{n+1}, k_1 + k_m)$ for $1 \leq m \leq j$ are linearly independent modulo $T(2^{n+1})$. There are $j + 1$ such elements, as required. As in (1) we observe

$$w(2^{n+1}, 2k_1) \equiv w(2^n, k_1)w(2^n, k_1) \mod T(2^{n+1}),$$

and similarly, for each $m \in \{1, \ldots, j\}$ we have

$$w(2^{n+1}, k_1 + k_m) \equiv w(2^n, k_1)w(2^n, k_m) + w(2^n, k_m)w(2^n, k_1) \mod T(2^{n+1}) + \sum_{\substack{1 \leq p < m \\ 1 \leq q < m}} \mathbb{K}w(2^n, k_p)w(2^n, k_q).$$

Therefore, $w(2^{n+1}, k_1+k_m)$ contains the summand $w(2^n, k_1)w(2^n, k_m)+w(2^n, k_m)w(2^n, k_1)$ which no $w(2^{n+1}, k_1 + k_p)$ with $p < m$ contains.
Finally,

$$w(2^{n+1}, 2k_j) \equiv w(2^n, k_j)w(2^n, k_j) \mod T(2^{n+1}) + \sum_{p=1}^{j-1} w(2^n, k_p)A(2^n) + A(2^n)w(2^n, k_p),$$

so $w(2^{n+1}, 2k_j)$ contains the summand $w(2^n, k_j)w(2^{n+1}, k_j)$ which none of the previous elements contains. It follows that the $j + 1$ elements we exhibited are linearly independent modulo $T(2^{n+1})$. □

Proof of Proposition 4.1. We adapt the proof of [13, Theorem 10] to show how the additional assumptions may be satisfied. In fact, (10) is already part of the construction.

Recall that the proof of [13, Theorem 10] constructs sets $U(2^{n+1})$ and $V(2^{n+1})$ by induction. The following cases are considered:

1. $n \in S$ and $n+1 \in S$.
2. $n \notin S$.
3. $n \in S$ and $n+1 \notin S$.

We modify cases 2 and 3, while not changing case 1, which we repeat for convenience of the reader:

Case 1: $\boldsymbol{n \in S}$ and $\boldsymbol{n+1 \in S}$. Define $U(2^{n+1}) = T(2^{n+1})$ and $V(2^{n+1}) = V(2^n)V(2^n)$. Conditions (6,7) certainly hold. If, by induction, Conditions (3,5) hold for $U(2^n)$ and $V(2^n)$, they hold for $U(2^{n+1})$ and $V(2^{n+1})$ as well. Moreover, $\dim_{\mathbb{K}} V(2^n) = (\dim_{\mathbb{K}} V(2^n))^2$, inductively satisfying Condition (2). Finally, Condition (9) follows directly from Lemma 4.3.

Case 2: $\boldsymbol{n \notin S}$. We begin as in the original argument: $\dim_{\mathbb{K}} V(2^n) = 2$, and is generated by monomials, by the inductive hypothesis. Let m_1, m_2 be the distinct monomials that generate $V(2^n)$. Then $V(2^n)V(2^n) = \mathbb{K}m_1m_1 + \mathbb{K}m_1m_2 + \mathbb{K}m_2m_1 + \mathbb{K}m_2m_2$. By Lemma 4.2, there exists $h \in \{1,2\}$ such that

$$\dim_{\mathbb{K}}(W(2^{n+1}) + T(2^{n+1}) + m_hV(2^n)) \geq \dim(T(2^{n+1}) + m_hV(2^n)) + 2.$$

Permuting m_1 and m_2 if necessary, we assume $h = 2$, and set

$$U(2^{n+1}) = T(2^{n+1}) + m_2V(2^n), \qquad V(2^{n+1}) = \mathbb{K}m_1m_1 + \mathbb{K}m_1m_2.$$

It is clear that Conditions (1,3,6,7,9) hold, and Condition (5) follows from

$$\begin{aligned} A(2^{n+1}) &= A(2^n)A(2^n) \\ &= U(2^n)U(2^n) \oplus U(2^n)V(2^n) \oplus V(2^n)U(2^n) \oplus m_1V(2^n) \oplus m_2V(2^n) \\ &= U(2^{n+1}) \oplus V(2^{n+1}). \end{aligned}$$

Case 3: $\boldsymbol{n \in S}$ and $\boldsymbol{n+1 \notin S}$. We begin as in the original argument: we have $n = e(i) - 1$ for some $i > 0$. By the inductive hypothesis, we have $\dim_{\mathbb{K}}(W(2^n) + T(2^n)) \geq \dim_{\mathbb{K}} T(2^n) + i + 1$. One more application of Lemma 4.3 gives

$$\dim_{\mathbb{K}}(W(2^{n+1}) + T(2^{n+1})) \geq \dim_{\mathbb{K}} T(2^{n+1}) + i + 2.$$

So as to treat simultaneously the cases $i \in Z$ and $i \notin Z$, we extend Condition (4) to all $i \in \mathbb{N}$ by taking $F_i = 0$ and $s = 0$ if $i \notin Z$.

We know that F_i has a basis $\{f_1, \dots, f_s\}$ for some $f_1, \dots, f_s \in A(2^{e(i)})$ and $s < i$. Write each f_j as $f_j = \bar{f}_j + g_j$ for $\bar{f}_j \in V(2^n)V(2^n)$ and $g_j \in T(2^{n+1})$. Since $V(2^n)V(2^n) \cap T(2^{n+1}) = 0$, this decomposition is unique.

Since $s < i$, there are elements $w_1, w_2 \in W(2^{e(i)})$ such that

$$(\mathbb{K}w_1 + \mathbb{K}w_2) \cap (T(2^{n+1}) + \mathbb{K}\bar{f}_1 + \ldots + \mathbb{K}\bar{f}_s) = 0.$$

Let P be a a linear $\mathbb{K}$-subspace of $V(2^n)V(2^n)$ maximal with the properties that $(\mathbb{K}w_1 + \mathbb{K}w_2) \cap (P + T(2^{n+1})) = 0$ and $\bar{f}_j \in P$ for all $j \in \{1, \dots, s\}$.

Observe that P has codimension 2 in $V(2^n)V(2^n)$. Since the monomials in $V(2^n)V(2^n)$ form a basis, there are two such monomials, say m_1 and m_2, that are linearly independent modulo P. Define then

$$V(2^{n+1}) = \mathbb{K}m_1 + \mathbb{K}m_2, \qquad U(2^{n+1}) = T(2^{n+1}) + P.$$

Conditions (5,6) are immediately satisfied. Since each polynomial $f_j = g_j + \bar{f}_j$ belongs to $U(2^{n+1})$, Condition (4) is satisfied as well.

To end the proof, observe now that $\{w_1, w_2\}$ are linearly independent modulo $U(2^{n+1})$, so $\dim_{\mathbb{K}}(\mathbb{K}w_1 + \mathbb{K}w_2 + U(2^{n+1})) = \dim_{\mathbb{K}} U(2^{n+1}) + 2$; this proves (9). □

5. Proof of Theorem 2.1

We present P as a quotient $\bar{A}/\mathscr{E}$ for a suitable ideal $\mathscr{E}$; we follow [13, page 844]. First, $\mathscr{E}$ is a graded ideal: $\mathscr{E} = \mathscr{E}(1) + \mathscr{E}(2) + \cdots$, so it suffices to define $\mathscr{E}(n)$ for all $n \in \mathbb{N}$. By definition, $\mathscr{E}(n)$ is the maximal subset of $A(n)$ such that, if $m \in \mathbb{N}$ be such that $2^m \leq n < 2^{m+1}$, then

$$A(j)\mathscr{E}(n)A(2^{m+2} - j - n) \subseteq U(2^{m+1})A(2^{m+1}) + A(2^{m+1})U(2^{m+1}) \tag{3}$$

for all $j \in \{0, \dots, 2^{m+2} - n\}$; or, more briefly, $(A\mathscr{E}A)(2^m) \subseteq T(2^m)$ for all $m \in \mathbb{N}$.

Theorem 5.1. *The subset $\mathscr{E}$ is an ideal in $\bar{A}$. Moreover, $P := \bar{A}/\mathscr{E}$ is radical, has Gelfand-Kirillov dimension two, is $\mathbb{N}$-graded and generated by two degree-1 elements, and $P[X]$ is not radical.*

Proof. By [13, Theorem 20], the GK-dimension of P is at most 2; it is in fact exactly 2, by Bergman's gap theorem [4]. Also, P is radical by [13, Theorem 24]. Moreover, $z \in U(1) = \mathscr{E}(1)$, so P is generated by the images of x and y in $\bar{A}/\mathscr{E}$.

Recall that X is a free indeterminate commuting with x and y. Consider $n \geq 2$. By Proposition 4.1, not all $w(2^n, i)$ belong to $U(2^n)$, so $(x+Xy)^{2^n} \notin U(2^n) \otimes \mathbb{K}[X]$, so $(x+Xy)^{2^{n-2}} \notin \mathscr{E}[X]$ by (3), so $(x+Xy)^{2^{n-2}} \neq 0$ in $P[X]$. Since n may be taken arbitrarily large, it follows that $x + Xy$ is not nilpotent.

If X be now declared to have degree 0, then $P[X]$ is a graded ring, and $x+Xy$ is homogeneous and not nilpotent. However, in a graded ring, a homogeneous element belongs to the Jacobson radical if and only if it is nilpotent; it therefore follows that $P[X]$ is not radical. □

6. Final remarks

The methods employed here depend crucially on the hypothesis that $\mathbb{K}$ is countable. We don't know if it there are finitely generated radical algebras of Gelfand-Kirillov dimension two over an uncountable field. By Amitsur's theorem, such algebras must be nil.

The argument in Theorem 5.1 requires us, in particular, to construct a ring P such that $P[X]$ is not graded nil. We do not know if P is nil; if so, this would be an improvement over [11], in which Smoktunowicz constructs a nil ring R such that $R[X]$ is not nil.

We note that, over any countable field, nil algebras of Gelfand-Kirillov dimension at most three were constructed by Lenagan, Smoktunowicz and Young [8, 9].

It remains an open problem whether there exist affine self-similar algebras satisfying the conditions of Corollary 2.4.

We are also unable to construct an algebra of quadratic growth (i.e. whose growth function is bounded by a polynomial of degree two). The algebras R constructed here do admit an upper bound on their growth of the form $\dim_{\mathbb{K}}(R(1)+\cdots+R(n)) \leq Cn^2 \log(n)^3$, see [13, Theorem 20].

We finally refer to Zelmanov's survey [14] for a wealth of similar problems.

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